Th 1 (Uniform Continuous Th). Let $A \subseteq \mathbb{R}$ be bounded, & closed ($x_0 \in A$ whenever $x_0 \in \mathbb{R}$ is the limit of some seq in A). Then $f: A \to \mathbb{R}$ is cts iff it is unif. cts.

Pf. $\Leftarrow$ Trivial

$\Rightarrow$ Use the B-W Th & seq. criterion for cts. (as done in the class).

Th 2. Let $\emptyset \neq A \subseteq \mathbb{R}$ & let $f: A \to \mathbb{R}$ be unif. cts. Then it maps any Cauchy seq (in A) to a Cauchy seq. (the converse is not true). [see Th 2* (b) below.]

Th 2*: Let $\emptyset \neq A \subseteq \mathbb{R}$ & $f: A \to \mathbb{R}$. Each of the following cases ((a), (b)) implies that f maps every Cauchy seq in A to a Cauchy seq:

(a) A is closed & f is cts.

(b) f is unif. cts.

proof (b). Suppose that f is unif. cts on A and let (x_n) be a seq in A & be Cauchy. We wish to show that $(f(x_n))$ is Cauchy. To do this, let $\varepsilon > 0$. Since f is unif. cts, $\exists\, \delta > 0$ s.t.

$$(*) \quad x, u \in A \ \& \ |x-u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon.$$

For this δ, $\exists\, N \in \mathbb{N}$ s.t

(#) $\quad |x_n - x_m| < \delta \quad \forall\ m, n \geq N.$

and it follows from (*) that

$$|f(x_n) - f(x_m)| < \varepsilon \quad \forall\ m, n \geq N$$

(noting $x_k \in A\ \forall k$). Then $(f(x_n))$ is indeed a Cauchy seq.

(a). Suppose A is closed & f cts. Let (x_n) be a seq in A. By Completeness of $\mathbb{R}$, $x_n \to x \in \mathbb{R}$ for some x. Since A is closed, this x belongs to A. Since f is cts at x it follows from the Seq. Criterion that $f(x_n) \to f(x)$ and hence $(f(x_n))$ is Cauchy (any conv. seq is Cauchy).

Th3 (Unif. Cts. Extension). Let $f : (a,b) \to \mathbb{R}$ be ($\subseteq \mathbb{R}$) unif. cts. Then it can be extended unif. continuously to $[a,b]$.

pf. Let $x_0 \in \{a, b\}$. Then $\exists$ a seq (x_n) in (a,b) convergent to x_0. Since (x_n) ($\subseteq (a,b)$) is Cauchy and f is unif. cts on (a,b) it follows from Th 2 that $(f(x_n))$ is Cauchy and hence converges: we denote the limit by $f(x_0)$. This is well-defined and f is then cts at x_0. To show this, it is sufficient (by seq criterion) to show that

$$\lim_n f(y_n) = f(x_0)$$

where (y_n) is another seq in (a,b) convergent to x_0.
We note that the "combined seq" (in (a,b))

$$x_1, y_1, x_2, y_2, x_3, y_3, \cdots$$

also converges to x_0 (so Cauchy) and hence (by Th 2 again)

$$f(x_1), f(y_1), f(x_2), f(y_2), f(x_3), f(y_3) \cdots$$

is also Cauchy and hence converges to a limit which should be equal to the limit of its subsequences, so to $f(x_0)$; consequently f is cts at x_0.

$$f(x_0) = \lim_n f(x_n) = \lim_n f(y_n).$$

Finally, by Th 1, the extended function is unif. cts on $[a,b]$.

Th 3^*. Let $\emptyset \neq A \subseteq \mathbb{R}$ and

$$\bar{A} := \{ l \in \mathbb{R} : l \text{ is the limit of a convergent seq in } A \}$$

(thus $A \subseteq \bar{A}$ — by looking constant seq). Then each unif. cts function on A can be extended unif. cts on $\bar{A}$.

proof. Ex. (Hint: similar as for Th 3).